

ALGEBRAIC DEFINITION OF HOLONOMY ON POISSON MANIFOLD

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ABSTRACT. We give an algebraic construction of connection on the symplectic leaves of Poisson manifold, introduced in [6]. This construction is suitable for the definition of the linearized holonomy on a regular symplectic foliation.

1. INTRODUCTION

If the rank of a Poisson structure on a smooth manifold M is constant, there are various tools in the theory of foliations that can be applied to study the corresponding symplectic foliation. But there are many even very simple degenerate Poisson structures that are singular, i.e, the rank of the corresponding contravariant tensor field is not constant. Therefore, some new methods are needed to handle this situation.

In this paper we give an algebraic coonstruction of the linear holonomy on a Poisson manifold. This construction is suitable for the case, when the Poisson structure is not regular and the necessary requirements for the Poisson structure, directly follow from the construction.

2. ALGEBRAIC DEFINITION OF BOTT CONNECTION

Let A be a commutative algebra and $Der(A)$ be the Lie algebra of derivations of A . As A is a commutative algebra, the space $Der(A)$ is a module over A . For a given ideal I in A let us define the following Lie subalgebras in $Der(A)$:

$$Der_I(A) = \{X \in Der(A) \mid X(I) \subset I\}$$

and its subalgebra

$$Der_I(A)_0 = \{Y \in Der(A) \mid Y(A) \subset I\}.$$

It is clear that both of them are A -submodules in $Der(A)$.

There is a canonical homomorphism

$$\rho : Der_I(A) \longrightarrow Der(A/I)$$

and as it follows from the definition, we have that $Der_I(A)_0 = \ker(\rho)$. When ρ is an epimorphism, the ideal I is said to be a *submanifold constraint ideal* (see [4]). For brevity we shall use the term *submanifold ideal*. In this case we have the isomorphism

$$Der(A/I) \cong \frac{Der_I(A)}{Der_I(A)_0}$$

Date: June 3, 2007.

The work was supported by the Georgian National Science Foundation under grant GNSF/ST-06/4-050.

We use the term *distribution* for any Lie subalgebra which is also an A -submodule of $Der(A)$.

Definition 2.1 (Integral of Distribution). An ideal I in A is said to be an *integral* for a given distribution D if $D \subset Der_I(A)$.

Let us denote the intersection $D \cap Der_I(A)_0$ by D_0 .

Definition 2.2 (Complete Integral). An ideal $I \subset A$ is said to be a *complete integral* for a given distribution $D \subset Der(A)$ if the following three conditions are satisfied

- (1) I is a submanifold ideal for A
- (2) I is an integral for D
- (3) $D/D_0 = \frac{Der_I(A)}{Der_I(A)_0} \quad (\cong Der(A/I) \text{ because of (1)})$

Let $I \cdot D = \{\sum a_i X_i \mid a_i \in I, X_i \in D\}$. It is clear that $I \cdot D$ is a subspace of D_0 .

Definition 2.3 (Regular Integral). A complete integral I for a given distribution $D \in Der(A)$ is said to be *regular* if $D_0 = I \cdot D$.

The above definition of a regular integral is in agreement with the following theorem from the classical theory of distributions

Theorem 2.4 (see [5]). *Let D be a completely integrable distribution on a smooth manifold M , and $x_0 \in M$ be a point where $\dim(D_{x_0}) = r$. Then the point x_0 has an open neighborhood U with coordinates (u_1, \dots, u_n) such that*

$$\left\{ \frac{\partial}{\partial u_i}(x) \mid i = 1, \dots, r \right\} \subset D_x, \quad \forall x \in U$$

It follows from this theorem that if x_0 is a regular point for D , then $\dim(D_{x_0})$ is locally maximal, and therefore, the set $\left\{ \frac{\partial}{\partial u_i} \mid i = 1, \dots, r \right\}$ is a local basis of the $C^\infty(M)$ -module D . If $N \subset M$ is the integral submanifold for D , passing through the point x_0 and $X \in D$ is such that $X|_N = 0$, then $X = \sum_{i=1}^r \varphi_i \frac{\partial}{\partial u_i}$, $\varphi_i \in I_N$, where I_N denotes the ideal of such smooth functions on M that vanish on N .

For a given ideal $I \in A$ let $\pi : A \longrightarrow A/I$ be the natural quotient map. Introduce the following A -modules

$$\Gamma(I) = \{X : A \longrightarrow A/I \mid X(ab) = X(a)\pi(b) + \pi(a)X(b)\}$$

and its submodule

$$\Gamma(I)_0 = \{Y \in \Gamma(I) \mid Y(I) = \{0\}\}$$

We call the quotient module

$$V(I) = \frac{\Gamma(I)}{\Gamma(I)_0}$$

the space of *transversal derivations*. It is a module over the quotient algebra A/I . Consider the following operation $[\cdot, \cdot] : Der_I(A) \times \Gamma(I) \longrightarrow \Gamma(I)$, defined as

$$[U, s] = \rho(U) \circ s - s \circ U, \quad U \in Der_I(A), \quad s \in \Gamma(I)$$

The submodule $\Gamma(I)_0$ is invariant for this operation and therefore we can reduce this operation to

$$[\cdot, \cdot] : Der_I(A) \times V(I) \longrightarrow V(I)$$

The following properties of this operation follow directly from the definition

$$(2.1) \quad \begin{aligned} (1) \quad & [aU, v] = a[U, v] \\ (2) \quad & [U, kv] = U(k)v + k[U, v] \end{aligned}$$

where $a \in A$, $k \in A/I$, $U \in \text{Der}_I(A)$ and $v \in V(I)$.

Definition 2.5 (Linear Connection). *Linear connection* for $I \subset A$ which is a *regular integral* for a distribution $D \in \text{Der}(A)$, is an action (covariant derivation) of the elements of Lie algebra $\text{Der}(A/I) \cong \frac{D}{I \cdot D}$ on the A/I -module $V(I)$, defined as

$$\nabla_X(v) = [\tilde{X}, v]$$

where $X \in \frac{D}{I \cdot D}$, $v \in V(I)$ and $\tilde{X} \in D$ is an extension of X : $\rho(\tilde{X}) = X$.

Independence from the extension \tilde{X} : if \tilde{X}_1 and \tilde{X}_2 in D are such that $\rho(\tilde{X}_1) = \rho(\tilde{X}_2)$ then we have

$$\begin{aligned} \tilde{X}_1 - \tilde{X}_2 \in I \cdot D & \Leftrightarrow \tilde{X}_1 - \tilde{X}_2 = \sum k_i U_i, \quad k_i \in I, \quad U_i \in D \Rightarrow \\ \Rightarrow [\tilde{X}_1 - \tilde{X}_2, v] &= [\sum k_i U_i, v] = \sum k_i [U_i, v] = 0 \in V(I) = \frac{\Gamma(I)}{\Gamma(I)_0} \end{aligned}$$

The properties of covariant derivation for ∇ easily follow from the properties of the operation $[\cdot, \cdot]$. In the case of a singular (i.e., not regular) integral, the definition of linear connection encounters the following problem: the subalgebra D_0 , in general, does not coincide with $I \cdot D$, i.e., not every $\tilde{X} \in D$ that vanishes on A/I is of the form $\tilde{X} = \sum k_i U_i$ with $k_i \in I$ and $U_i \in D$. Hence the independence of the expression $[\tilde{X}, v]$ from the extension \tilde{X} is problematic. A Poisson structure helps to somehow resolve the difficulty. Some authors (see [6]) define the covariant derivation of a transversal vector field not by tangent vectors but by 1-forms. In the next section we shall review this construction in the case of a Poisson manifold.

3. LINEAR CONNECTION ON A SYMPLECTIC LEAF OF POISSON MANIFOLD

Let $(M, \{\cdot, \cdot\})$ be a Poisson manifold. For any function $f \in C^\infty(M)$, we denote by X_f the Hamiltonian vector field on M corresponding to f . The antisymmetric contravariant tensor (bivector) field on M corresponding to the Poisson structure, defines a homomorphism of the vector bundles $T^*(M) \rightarrow T(M)$. We denote the tangent vector corresponding to the cotangent vector $\alpha \in T^*(M)$ also by X_α . For a differential 1-form ω we have a vector field X_ω defined by the above homomorphism.

Let $N \subset M$ be a symplectic leaf. Denote by $T(N)^\perp$ the subbundle of the restricted bundle $T^*(M)|_N$:

$$T(N)^\perp = \{\omega \in T^*(M) \mid x \in N, \omega(T_x(N)) = 0\}$$

Linear connection on the vector bundle $T(N)^\perp$ is a covariant derivation of the sections of this bundle by the sections of the vector bundle $T^*(M)|_N$ defined as follows

$$\nabla_\alpha(\omega) = L_{X_{\tilde{\alpha}}}(\tilde{\omega})|_N$$

where:

- α is a section of $T^*(M)|_N$ and $\tilde{\alpha}$ is its extension (at least local) on M : $\tilde{\alpha}|_N = \alpha$;

- ω is a section of $T(N)^\perp$ and $\tilde{\omega}$ is its extension (at least local) on M : $\tilde{\omega}|_N = \omega$;
- $L(\cdot)$ is the operation of Lie derivation.

If $\tilde{\alpha} = \varphi \cdot \tilde{\alpha}_1$, where $\varphi \in C^\infty(M)$ and $\tilde{\alpha}_1$ is a 1-form, then by definition of the Lie derivation we have

$$L_{X_{\tilde{\alpha}}}(\tilde{\omega})|_N = (d\varphi \cdot \tilde{\omega}(X_{\tilde{\alpha}_1}))|_N + (\varphi \cdot L_{X_{\tilde{\alpha}_1}}(\tilde{\omega}))|_N$$

The first term of the RH side is 0 because the vector field $X_{\tilde{\alpha}_1}|_N$ is always tangent to the symplectic leaf N , and $\tilde{\omega}|_N = \omega$ is a section of $T(N)^\perp$. Therefore we have

$$L_{X_{\tilde{\alpha}_1}}(\tilde{\omega})|_N = \varphi|_N \cdot L_{X_{\tilde{\alpha}_1}}(\tilde{\omega})|_N$$

from which easily follows as the independence of the expression $L_{X_{\tilde{\alpha}}}(\tilde{\omega})|_N$ from the extension $\tilde{\alpha}$ so the $C^\infty(N)$ -linearity of $\nabla_\alpha(\omega)$ by the argument α .

For $\tilde{\omega} = \varphi \cdot \tilde{\omega}_1$, we have

$$(3.1) \quad L_{X_{\tilde{\alpha}}}(\varphi \cdot \tilde{\omega}_1)|_N = (X_{\tilde{\alpha}}(\varphi) \cdot \tilde{\omega}_1)|_N + (\varphi \cdot L_{X_{\tilde{\alpha}}}(\tilde{\omega}_1))|_N$$

From this we obtain the independence from the extension $\tilde{\omega}$ in the following way: if $\tilde{\omega}|_N = \tilde{\omega}'|_N$, then $\tilde{\omega} - \tilde{\omega}' = \sum \varphi_i \tilde{\omega}_i$, where $\varphi_i|_N = 0$; then recall that as $X_{\tilde{\alpha}}|_N$ is tangent to N , we have that $X_{\tilde{\alpha}}(\varphi_i)|_N = 0$. From the equality 3.1 also follows the Leibniz rule for the argument ω in $\nabla_\alpha(\omega)$.

Let us describe the dual definition of linear connection, which is more compatible with the algebraic definition introduced in the previous section. In this case, instead of the vector bundle $T(N)^\perp$, we take the quotient bundle $\frac{T(M)|_N}{T(N)} \equiv V(N)$, which is the bundle of dual spaces of the fibers of $T(N)^\perp$. For a section α of $T^*(M)|_N$ and a section s of $V(N)$, we set

$$(3.2) \quad \nabla_\alpha(s) = \rho([X_{\tilde{\alpha}}, \tilde{s}])$$

where: $\tilde{\alpha}$ is an extension of α as in the previous definition; \tilde{s} is a vector field on M , such that $\rho(\tilde{s}|_N) = s$ and $\rho : T(M)|_N \rightarrow V(N)$ is the quotient map. The independence of the expression 3.2 from the extensions $\tilde{\alpha}$ and \tilde{s} , as the properties of covariant derivation easily follow from the definition by analogy to the case of the dual definition.

In general, if ∇ is a covariant derivation on some vector bundle, its dual on the dual vector bundle is defined by the equality

$$(3.3) \quad \langle \nabla_\alpha(\omega), s \rangle = \alpha \langle \omega, s \rangle - \langle \omega, \nabla_\alpha(s) \rangle$$

where ω and s are sections of the bundles dual to each other. Keeping in mind this equality and follow the definitions of the linear connections on $T(N)^\perp$ and its dual $V(N)$, we obtain

$$L_{X_{\tilde{\alpha}}}(\tilde{\omega})(\tilde{s}) = \tilde{s}(\tilde{\omega}(X_{\tilde{\alpha}})) + X_{\tilde{\alpha}}(\tilde{\omega}(\tilde{s})) - \tilde{s}(\tilde{\omega}(X_{\tilde{\alpha}})) - \tilde{\omega}([X_{\tilde{\alpha}}, \tilde{s}]) = X_{\tilde{\alpha}}(\tilde{\omega}(\tilde{s})) - \tilde{\omega}([X_{\tilde{\alpha}}, \tilde{s}])$$

from which follows the equality 3.3 for ω – a section of $T(N)^\perp$, s – a section of $V(N)$ and $\alpha \in T^*(M)|_N$.

Let us consider an example which is rather simple but useful for the demonstration of various interesting properties of singular Poisson structures.

Example 1. Let S be a 2-dimensional symplectic manifold, with Poisson bracket $\{ , \}$. Let φ be a smooth function on S , such that $\varphi^{-1}(0) = \{x_0\}$. Consider a modified Poisson bracket $\{ , \}_1 = \varphi \cdot \{ , \}$ (as the manifold is 2-dimensional this bracket satisfies all the properties required for a Poisson bracket). The symplectic foliation for the modified bracket consists of two symplectic leaves: $\{x_0\}$ and $S \setminus$

$\{x_0\}$. A fiber of the transversal foliation for the leaf $S \setminus \{x_0\}$ is just $\{0\}$. Hence it is more interesting to consider the linear connection for the leaf consisting of just one point $\{x_0\}$. In this case the transversal space is $V(x_0) = T_{x_0}(S)$ and thus, by definition, the covariant derivation, corresponding to the linear connection, is an action of the elements of $T_{x_0}^*(S)$ on $T_{x_0}(S)$:

$$(3.4) \quad \nabla_\alpha(V) = [X_f^1, \tilde{V}]|_{x_0} = [\varphi \cdot X_f, \tilde{V}]|_{x_0} = V(\varphi) \cdot X_f(x_0)$$

where $f \in C^\infty(S)$ is such that $df(x_0) = \alpha$; X_f^1 is the Hamiltonian vector field for f with respect to the modified bracket $\{, \}_1$ and X_f is the Hamiltonian vector field for f with respect to the original bracket $\{, \}$. Notice that as the original bracket is nondegenerate, we have that $\alpha \neq 0 \Leftrightarrow X_\alpha \neq 0$, therefore the equation $\nabla_\alpha(V) = 0$, for V , when $\alpha \neq 0$ is equivalent to $V \in \ker(\varphi'(x_0))$. In other words, the “flat” sections of the vector bundle $V(x_0)$ over $\{x_0\}$ are the elements of $\ker(\varphi'(x_0))$. Also it follows from the equality 3.4 that the linear connection on the leaf $\{x_0\}$ is flat if and only if $\varphi'(x_0) = 0$.

4. LINEAR CONNECTION FOR TRANSVERSAL MULIVECTORS AND SCHOUTEN-NIJENHUIS BRACKET

For an arbitrary smooth manifold M we denote by $\mathcal{V}^k(M)$, $k = 0, \dots, \infty$, the space of skew-symmetric contravariant tensor (multivector) fields of degree k on M ($\mathcal{V}^0(M) = C^\infty(M)$). The Schouten-Nijenhuis bracket is a linear operation

$$[\cdot, \cdot] : \mathcal{V}^p(M) \times \mathcal{V}^q(M) \longrightarrow \mathcal{V}^{p+q-1}(M)$$

with the following properties (see [8])

$$(4.1) \quad \begin{aligned} (1) \quad & [U, V] = (-1)^{|U| \cdot |V|} \cdot [V, U] \\ (2) \quad & [U, V \wedge W] = [U, V] \wedge W + (-1)^{(|U|+1) \cdot |V|} \cdot V \wedge [U, W] \\ (3) \quad & (-1)^{|U| \cdot (|W|-1)} \cdot [U, [V, W]] + (-1)^{|V| \cdot (|U|-1)} \cdot [V, [W, U]] + \\ & + (-1)^{|W| \cdot (|V|-1)} \cdot [W, [U, V]] = 0 \end{aligned}$$

For monomial type multivector fields the formula for the Schoute-Nijenhuis bracket is

$$(4.2) \quad [X_1 \wedge \dots \wedge X_m, Y_1 \wedge \dots \wedge Y_n] =$$

$$(-1)^{m+1} \sum_{i,j} (-1)^{i+j} [X_i, Y_j] \wedge X_1 \wedge \dots \wedge \hat{X}_i \wedge \dots \wedge X_m \wedge Y_1 \wedge \dots \wedge \hat{Y}_j \wedge \dots \wedge Y_n$$

If we define a generalization of the ordinary Lie derivation for the multivector fields as

$$L_X = i_X \circ d - (-1)^{|X|} d \circ i_X$$

then there is the following relation between this operation and the Schouten-Nijenhuis bracket

$$(4.3) \quad [L_X, i_Y] = i_{[X, Y]}$$

where X and Y are multivector fields and i denotes the inner product of a multivector field and a differential form.

For an arbitrary vector bundle E and a covariant derivation ∇ on it, the canonical extension of ∇ to the Grassmann algebra bundle $\wedge(E) = \sum_{k=0}^{\infty} \wedge^k(E)$ is defined by

the formula

$$(4.4) \quad \nabla_X(S_1 \wedge \cdots \wedge S_k) = \sum (-1)^{i+1} \nabla_X(S_i) \wedge S_1 \wedge \cdots \wedge \hat{S}_i \wedge \cdots \wedge S_k$$

where $S_1 \wedge \cdots \wedge S_k$ is a section of $\wedge^k(E)$.

For a symplectic manifold M and a symplectic leaf $N \subset M$, the k -th exterior degree of the vector bundle $V(N) = \frac{T(M)|_N}{T(N)}$ is canonically isomorphic to $\frac{\wedge^k T(M)|_N}{T(N) \wedge (\wedge^{k-1} T(M))|_N}$. Here $T(N) \wedge (\wedge^{k-1} T(M))|_N$ is the intersection of the ideal generated by $T(N)$ in the Grassmann algebra bundle $\wedge T(M)|_N$, with $\wedge^k T(M)|_N$. According to the formulas 4.2 and 4.4, the extension of the Bott covariant derivation on $V(N)$ to $\wedge^k V(N)$ can be defined as

$$(4.5) \quad \nabla_\alpha(U) = \rho([X_{\tilde{\alpha}}, \tilde{U}]|_N)$$

where α is a section of $T^*(M)|_N$; U is a section of $\wedge^k V(N)$;

$$\rho : \wedge^k T(M)|_N \longrightarrow \wedge^k V(N)$$

is the quotient map; $\tilde{\alpha}$ is an extension of α ; \tilde{U} is such a section of $\wedge^k T(M)$ (i.e., a multivector field on M) that $\rho(\tilde{U}|_N) = U$ and $[\ , \]$ is the *Schouten-Nijenhuis Bracket*.

Example 2. Let $S, \{ \ , \ }_1 = \varphi\{ \ , \ }$ and x_0 be the same objects as in the Example 1. We have that $\wedge^2 V(x_0) = \wedge^2 T_{x_0}(S)$. For $\alpha \in T_{x_0}^*(S)$ and $U \wedge V \in T_{x_0}(S)$, we have

$$\nabla_\alpha(U \wedge V) = \nabla_\alpha(U) \wedge V - \nabla_\alpha(V) \wedge U =$$

$$U(\varphi) \cdot X_\alpha \wedge V - V(\varphi) \cdot X_\alpha \wedge U = X_\alpha \wedge (U(\varphi) \cdot V - V(\varphi) \cdot U)$$

After this suppose that we need to solve the equation $\nabla_\alpha(U \wedge V) = 0$ for $U \wedge V$. As $\dim(S) = 2$, the space $\wedge^2 T_{x_0}(S)$ is 1-dimensional. Therefore we are free to take the vector V as an element of $\ker(\varphi'(x_0))$. Hence we obtain

$$\nabla_\alpha(U \wedge V) = U(\varphi) \cdot X_\alpha \wedge V = 0$$

From this follows that if $\varphi'(x_0) = 0$, then the extended linear connection on $\wedge^2 T_{x_0}(S)$ is flat, otherwise, the equation $\nabla_\alpha(U \wedge V) = 0$ has a nontrivial solution if and only if $X_\alpha \in \ker(\varphi'(x_0))$. The both case are summarized as the following: the equation $\nabla_\alpha(U \wedge V) = 0$ for has nontrivial solution if and only if $X_\alpha \in \ker(\varphi'(x_0))$, and the set of solutions is $U \wedge \ker(\varphi'(x_0))$ (of course $\cong \wedge^2 T_{x_0}(S)$).

5. LIE ALGEBRA COHOMOLOGIES WITH VALUES IN A MODULE

We start from review of some definitions and facts from the cohomology theory of Lie algebras. Let L be a Lie algebra, A be an associative and commutative algebra over the field of complex (or real) numbers and S be a module over the algebra A .

Definition 5.1. The triple (L, S, A) is said to be a Lie module over the Lie algebra L if the following conditions are satisfied:

- (11) L is a module over the algebra A and there is a Lie algebra homomorphism $\phi : L \longrightarrow \text{Der}(A)$, which is also a homomorphism of A -modules, such that for any $a \in A$ and $X, Y \in L$, we have: $[X, aY] = X(a)Y + a[X, Y]$. Here $\text{Der}(A)$ denotes the space of derivations of the algebra A and $X(a)$ denotes $\phi(X)(a)$.

- (12) There is a Lie algebra homomorphism $\psi : L \longrightarrow \text{End}_{\mathbb{C}}(S)$, which is also a homomorphism of A -modules, such that for any $X \in L$, $s \in S$ and $a \in A$, we have: $X(as) = X(a)s + aX(s)$. Here $\text{End}_{\mathbb{C}}(S)$ denotes the space of \mathbb{C} -linear mappings from S to itself and $X(s)$ denotes $\psi(X)(s)$.

Sometimes, for brevity, the term “Lie module over L ” will be used for the A -module S .

For any integer number $m \geq 1$, let us denote the space of A -multilinear mappings from $\underbrace{L \times \cdots \times L}_{m\text{-times}}$ to the Lie module S , by $\Omega_A^m(L, S)$. We also set that $\Omega_A^0(L, S) = S$

and $\Omega_A(L, S) = \bigoplus_{m=0}^{\infty} \Omega_A^m(L, S)$.

For any $\omega \in \Omega_A^m(L, S)$ and $\{X_1, \dots, X_{m+1}\} \subset L$, define $d\omega$ by the well-known Koszul formula:

$$\begin{aligned} (d\omega)(X_1, \dots, X_{m+1}) &= \sum_i (-1)^{i-1} X_i \omega(X_1, \dots, \widehat{X_i}, \dots, X_{m+1}) + \\ &+ \sum_{i < j} (-1)^{i+j} \omega([X_i, X_j], \dots, \widehat{X_i}, \dots, \widehat{X_j}, \dots) \end{aligned}$$

Lemma 5.2. *If the A -module L is projective then the following three conditions are equivalent:*

- (1) *For any $a \in A$ and $s \in \Omega_A^0(L, S) = S$: $d(as) = d(a)s + ad(s)$.*
- (2) *The A -module S is a Lie module over L .*
- (3) *For any integer $m \geq 1$ and $\omega \in \Omega_A^m(L, S)$, the mapping*

$$d\omega : L^{m+1} \longrightarrow S$$

is A -multilinear (i.e., $(d\omega)(aX_1, \dots, X_{m+1}) = a(d\omega)(X_1, \dots, X_{m+1})$).

Proof. It is clear that the conditions 1 and 2 are equivalent, because for any $X \in L$, we have that:

$$(d(as))(X) = X(as) = X(a)s + aX(s) = (d(a)s + ad(s))(X).$$

Also, it can be verified by direct calculation that from the condition 2 (or, which is the same, from 1) follows the condition 3. Now, suppose that the condition 3 is true. In this case For any $X, Y \in L$, $a \in A$ and $\omega \in \Omega_A^1(L, S)$, we have:

$$\begin{aligned} (d\omega)(X, aY) &= a(d\omega)(X, Y) \quad \Rightarrow \\ X\omega(aY) - aY\omega(X) - \omega([X, aY]) &= aX\omega(Y) - aY\omega(X) - a\omega([X, Y]) \Rightarrow \\ X(a\omega(Y)) - aY\omega(X) - X(a)\omega(Y) - a\omega([X, Y]) &= \\ = aX\omega(Y) - aY\omega(X) - a\omega([X, Y]) \quad \Rightarrow \\ X(a\omega(Y)) &= X(a)\omega(Y) + aX(\omega(Y)) \end{aligned}$$

Because it is assumed that the A -module L is projective, for any $s \in S$ can be found such $\omega \in \Omega_A^1(L, S)$ and $Y \in L$, that $\omega(Y) = s$. Therefore, we obtain that $X(as) = X(a)s + aX(s)$, i.e., S is a Lie module over the Lie algebra L . ■

Hence, we have that if the A -module S is a Lie module over the Lie algebra L , the operator d , carries the space $\Omega_A^m(L, S)$ into $\Omega_A^{m+1}(L, S)$, which implies that the pair $(\Omega_A(L, S), d)$ is a differential complex.

6. LIE ALGEBRA HOMOLOGIES AND SUPERCOMMUTATOR

Let \mathfrak{g} be a Lie algebra and a module over a commutative and associative algebra B . If the Lie algebra bracket in \mathfrak{g} is bilinear for the elements of B ($[x, ay] = a[x, y]$, $\forall x, y \in \mathfrak{g}$ and $a \in B$) one has a well-defined boundary operator $\delta : \wedge_B^m(\mathfrak{g}) \longrightarrow \wedge_B^{m-1}(\mathfrak{g})$, where \wedge_B denotes the exterior product of a B -module by itself:

$$\delta(x_1 \wedge \cdots \wedge x_m) = \sum_{i < j} (-1)^{i+j} [x_i, x_j] \wedge \cdots \wedge \widehat{x_i} \wedge \cdots \wedge \widehat{x_j} \wedge \cdots \wedge x_m, \quad m > 1$$

and $\delta(x) = 0, \forall \{x, x_1, \dots, x_m\} \subset \mathfrak{g}$

The homology given by the boundary operator δ is known as the homology of the Lie algebra \mathfrak{g} with coefficients from the algebra B .

There is a useful relation between the coboundary operator δ and the Schouten-Hijenhuis bracket on the exterior algebra $\wedge_B(\mathfrak{g})$: $\forall u \in \wedge_B^m(\mathfrak{g})$ and $\forall v \in \wedge_B(\mathfrak{g})$, we have

$$(6.1) \quad [u, v] = \delta(u) \wedge v + (-1)^m u \wedge \delta(v) - \delta(u \wedge v)$$

It can be said that the supercommutator $[\cdot, \cdot]$ measures the deviation of the boundary operator δ from being an antiderivative of degree -1 (see [?]). From the formula 6.1, easily follows that the induced supercommutator on the homology space $H_\bullet(\mathfrak{g}, B)$ is trivial: for $u \in \wedge_B^m \mathfrak{g}$ and $v \in \wedge_B \mathfrak{g}$, such that $\delta(u) = \delta(v) = 0$, we have $[u, v] = -\delta(u \wedge v)$, which implies that the homology class of the element $[u, v]$ is trivial.

From the formula 6.1, also follows that the homology space $H_\bullet(\mathfrak{g}, B)$ does not inherits the exterior algebra structure from $\wedge_B(\mathfrak{g})$: if the elements $u, v \in \wedge_B(\mathfrak{g})$ are closed, then we have that $\delta(u \wedge v) = -[u, v]$.

7. THE HOMOLOGY SPACE OF A LIE ALGEBRA IDEAL

Let L be a Lie algebra and A be a commutative and associative algebra over \mathbb{C} (or \mathbb{R}). Suppose that the pair (L, A) satisfies the condition $l1$ from the definition 5.1.

Let $V \subset L$ be an ideal in the Lie algebra L (i.e., $\forall v \in V$ and $\forall x \in L$: $[v, x] \in V$). Define the subalgebra A_V , in the algebra A , as

$$A_V = \{a \in A \mid v(a) = 0, \forall v \in V\}$$

and the subspace V' in L , as

$$V' = \left\{ \sum_{i=1}^n a_i v_i \mid n \in \mathbb{N} \ a_i \in A_V, \ v_i \in V, \ i = 1, \dots, n \right\}$$

The latter can be defined as the minimal A_V -submodule of L , containing V .

For any $x \in L$, $v \in V$ and $a \in A_V$, we have

$$v(x(a)) = [v, x](a) + x(v(a)) = 0 \quad \Rightarrow \quad x(a) \in A_V$$

which implies that the subalgebra A_V is invariant under the action of the elements of L . Therefore, we have that $[x, av] = x(a)v + a[x, v] \in V'$, which means that V' is also an ideal in the Lie algebra L . We call the ideal V' the *complement* of the ideal V .

From the fact that V is a subspace of V' follows that $A_{V'} \subset A_V$, but on the other hand, for any $a, b \in A_V$ and $v \in V$, we have that $(bv)(a) = b \cdot v(a) = 0$, which

means that any $a \in A_V$ is also an element of $A_{V'}$. Hence, the two algebras A_V and $A_{V'}$, coincide.

The ideal V in the Lie algebra L will be called *complete*, if $V = V'$. Further, by default, we assume that the ideal V is complete (i.e., V is a module over the algebra A_V).

Consider the homology space of the Lie algebra V with coefficients from A_V . One can define an action of the Lie algebra L on the A_V -module $H_\bullet(V, A_V)$: for $X \in L$ and $v \in V$ such that $\delta(v) = 0$, let $X([v]) = [[X, v]]$, where $[\cdot]$ denotes the homology class and $[\cdot, \cdot]$ denotes the supercommutator.

Proposition 7.1. *The action of the Lie algebra L on the homology space $H_\bullet(V, A_V)$ is correctly defined and gives a Lie module structure on the A_V -module $H_\bullet(V, A_V)$.*

Proof. As it follows from the formula 6.1, we have that for any $X \in L$ and a closed element $u \in \wedge_{A_V}(V)$: $[X, u] = -\delta(X \wedge u)$, which implies that $[X, u]$ is closed. If the element u is exact and $u = \delta(v)$, then we have

$$\delta([X, v]) = \delta(-X \wedge \delta(v) - \delta(X \wedge v)) = -\delta(X \wedge u) = -[X, u]$$

which implies that the element $[X, u]$ is exact. From these two fact follows that the closed and exact element in the exterior algebra $\wedge_{A_V}(V)$ are invariant under the supercommutator with the elements of the Lie algebra L . Therefore, the action, $X([u]) = [[X, u]]$, of L on the homology space $H_\bullet(V, A_V)$ is correctly defined. By definition of the algebra A_V and the properties of the supercommutator, easily follow that for any $a \in A_V$: $(aX)([u]) = aX([u])$ and $X(a[u]) = X([au]) = [[X, au]] = X(a)[u] + aX([u])$; which implies that the A_V -module $H_\bullet(V, A_V)$ is a Lie module over the Lie algebra L . ■

It follows from the above proposition, that for each integer $n \geq 0$, one can consider the the following differential complex $(\Omega_{A_V}(L, H_n(V, A_V)), d)$, which gives the Lie algebra cohomology of L with values in the A_V -module $H_n(V, A_V)$.

As it was mentioned early, the induced supercommutator on each homology space $H_n(V, A_V)$, $n = 0, \dots, \infty$, is trivial. Therefore, the action of the Lie algebra V on each $H_n(V, A_V)$ is trivial, which implies that the submodule of the A_V -module $\Omega_{A_V}(L, H_\bullet(V, A_V))$, consisting of such forms ω , that $i_v(\omega) = 0$, $\forall v \in V$, is invariant under the action of the differential d ($i_v\omega = 0 \Rightarrow i_v(d\omega) = 0$, $\forall v \in V$). Hence, one can consider the subcomplex $(\Omega_{A_V}(L, H_\bullet(V, A_V))_0, d)$ of the differential complex $(\Omega_{A_V}(L, H_n(V, A_V)), d)$, where

$$\Omega_{A_V}(L, H_\bullet(V, A_V))_0 = \{\omega \in \Omega_{A_V}(L, H_\bullet(V, A_V)) \mid i_v(\omega) = 0, \forall v \in V\}$$

Actually, the complex $(\Omega_{A_V}(L, H_\bullet(V, A_V))_0, d)$ is canonically isomorphic to the complex $(\Omega_{A_V}(L/V, H_\bullet(V, A_V)), d)$.

8. THE CHARACTERISTIC CLASS OF A LIE ALGEBRA IDEAL

Let us denote by $\Omega_{A_V}(L, H_\bullet(V, A_V))_1$ the quotient module

$$\Omega_{A_V}(L, H_\bullet(V, A_V)) / \Omega_{A_V}(L, H_\bullet(V, A_V))_0$$

For any integer $n \geq 0$, we have the following short exact sequence:

$$\begin{aligned} 0 &\longrightarrow \Omega_{A_V}(L/V, H_n(V, A_V)) \longrightarrow \Omega_{A_V}(L, H_n(V, A_V)) \xrightarrow{\pi} \\ &\xrightarrow{\pi} \Omega_{A_V}(L, H_n(V, A_V))_1 \longrightarrow 0 \end{aligned}$$

which induces the standard homomorphism of cohomology spaces

$$\sigma_n : H^\bullet(L, H_n(V, A_V))_1 \longrightarrow H^{\bullet+1}(L/V, H_n(V, A_V))$$

where $H^\bullet(L, H_n(V, A_V))_1$ denotes the cohomology space of the complex $(\Omega_{A_V}(L, H_n(V, A_V)), d)$ and σ_n is defined as: $\sigma_n([\pi(\omega)]) = [d\omega]$, for $\omega \in \Omega_{A_V}(L, H_n(V, A_V))$, such that $d(\pi(\omega)) = 0$ ($\Rightarrow \pi(d\omega) = 0 \Rightarrow d\omega \in \Omega_{A_V}(L/V, H_n(V, A_V))$); here $[d\omega]$ denotes the cohomology class of the form $d\omega$ in $H^{\bullet+1}(L/V, H_n(V, A_V))$.

Consider the special case when $n = 1$. Let us denote the homology space $H_1(V, A_V)$ by H_V . So, we have an exact sequence

$$(8.1) \quad 0 \longrightarrow \Omega_{A_V}^\bullet(L/V, H_V) \longrightarrow \Omega_{A_V}^\bullet(L, H_V) \xrightarrow{\pi} \Omega_{A_V}^\bullet(L, H_V)_1 \longrightarrow 0$$

and the homomorphism: $\sigma_1 : H^\bullet(L, H_V)_1 \longrightarrow H^{\bullet+1}(L/V, H_V)$.

There is a one-to-one correspondence between the set of splittings of the following short exact sequence

$$0 \longrightarrow V \longrightarrow L \xrightarrow{p} L/V \longrightarrow 0$$

and the set of such homomorphisms $\alpha : L \longrightarrow V$, that $\alpha(v) = v$, $\forall v \in V$ (projection operator). Any such projection operator α , defines a 1-form $\tilde{\alpha} \in \Omega_{A_V}^1(L, H_V)$:

$$\tilde{\alpha}(X) = [\alpha(X)], \forall X \in L$$

Lemma 8.1. *For any two projections $\alpha, \beta : L \longrightarrow V$, the forms $\pi(\tilde{\alpha})$ and $\pi(\tilde{\beta})$ in $\Omega_{A_V}^1(L, H_V)_1$, are equal and closed.*

Proof. For such α and β we have: $(\alpha - \beta)(v) = v - v = 0$, $\forall v \in V$, which implies that $i_v(\tilde{\alpha} - \tilde{\beta}) = 0 \Rightarrow \tilde{\alpha} - \tilde{\beta} \in \Omega_{A_V}^1(L/V, H_V) \Rightarrow \pi(\tilde{\alpha}) = \pi(\tilde{\beta})$. For any $v \in V$ and $X \in L$, we have:

$$\begin{aligned} (d\tilde{\alpha})(v, X) &= v(\tilde{\alpha}(X)) - X(\tilde{\alpha}(v)) - \tilde{\alpha}([v, X]) = \\ &= -X([v]) - [[v, X]] = -[[X, v]] - [[v, X]] = 0 \end{aligned}$$

Therefore, we obtain that $i_v(d\tilde{\alpha}) = 0$, $\forall v \in V$, which means that the form $d\tilde{\alpha}$ belongs to the submodule $\Omega_{A_V}^2(L/V, H_V)$, or equivalently: $\pi(d\tilde{\alpha}) = d\pi(\tilde{\alpha}) = 0 \Rightarrow \pi(\tilde{\alpha})$ is closed in $\Omega_{A_V}^1(L, H_V)_1$.

Remark 1. $d\tilde{\alpha} - d\tilde{\beta} = d(\tilde{\alpha} - \tilde{\beta}) \Rightarrow$ the forms $\tilde{\alpha}$ and $\tilde{\beta}$ are cohomological in $\Omega_{A_V}^2(L/V, H_V)$, and their cohomology class is exactly $\sigma_1(\pi(\tilde{\alpha})) \in H^2(L/V, H_1(V, A_V))$.

■

As it follows from the above lemma, for any projection homomorphism $\alpha : L \longrightarrow V$, the element $\pi(\tilde{\alpha})$ in $\Omega_{A_V}^1(L, H_V)_1$ is one and the same. Let us denote this element (and also its cohomology class in $H^1(L, H_V)_1$) by Δ . The element $\sigma_1(\Delta)$ in $H^2(L/V, H_1(V, A_V))$ we call the *characteristic class* of the Lie algebra ideal V in L .

Remark 2. It is clear that if V is an ideal in the Lie algebra L , then the space $[V, V] = \left\{ \sum_i a_i [u_i, v_i] \mid a_i \in A_V, u_i, v_i \in V \right\}$ is also an ideal in L (an in V , too).

We can consider the Lie algebra $\tilde{L} = L/[V, V]$ and its commutative subalgebra $\tilde{V} = V/[V, V]$, which is also an ideal in \tilde{L} . It is clear that the homology space

$H_1(V, A_V)$ is the same as \tilde{V} and the quotient \tilde{L}/\tilde{V} is the same as L/V . One can consider the following short exact sequence

$$(8.2) \quad 0 \longrightarrow \tilde{V} \longrightarrow \tilde{L} \longrightarrow L/V \longrightarrow 0$$

As the Lie algebra \tilde{V} is commutative, in this situation, for any connection form $\alpha : \tilde{L} \longrightarrow \tilde{V}$ (a splitting of the short exact sequence 8.2), its curvature form coincides with $d\alpha \in \Omega_{A_V}^2(L/V, \tilde{V})$, and the cohomology class of the form $d\alpha$ in $d\alpha \in \Omega_{A_V}^2(L/V, \tilde{V})$ coincides with the characteristic class of the ideal V . Therefore, to study the properties of the characteristic class of a Lie algebra ideal, we can consider the case when the ideal is commutative.

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